Towards Predicate Fuzzy Partial Logic

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Fuzzy logic [5], suitable for handling gradual truth, and partial logic [4], dealing with undefined truth, have arguably orthogonal motivations; thus it is meaningful to consider undefined truth in the context of gradual propositions. The resulting fuzzy partial logic should provide a framework in which some gradual propositions may fail to possess any truth degree.

A simple system of *propositional* fuzzy partial logic has recently been proposed by Běhounek and Novák [3]. This system renders undefined truth by means of a dummy truth value $*$, added to each L-algebra of truth degrees for the underlying fuzzy logic L (assumed to be an implicative [6] expansion of the logic MTL_Δ). The connectives of L then need be extended to operate on $*$, too; this can be done in several meaningful ways, including Bochvar-style (where $*$ acts as annihilator), Sobociński-style (where $*$ acts as the neutral element), Kleene-style (which keeps the annihilators of the original L-algebra), Bochvar-external style (which treats $*$ as 0), interval-valued style, McCarthy sequential style, etc. The propositional fuzzy partial logic L^* introduced in [3] makes all of these (and further) variants available as defined connectives (with only a handful of them taken as primitive). The notions of tautologicity and entailment for L^* are derived as usual from the intended algebraic semantics, with 1 (the full defined truth) regarded as the only designated truth value. An axiomatic system for L^* (extending the rules of L, suitably adapted to handle $*$) has been proposed in [3], which is sound and complete w.r.t. general, linear, and (if L enjoys standard completeness) standard L-algebras expanded by $*$.

A natural next step is to extend the propositional fuzzy partial logic L^* to its first- and higher-order variants. In this paper, we propose first- and higher-order variants of L^* that accommodate undefined truth (represented by the dummy truth value $*$), but leave aside undefined individuals (which in a more general setting might result, e.g., from undefined values of mathematical operations, non-denoting ι-terms, etc.).

The semantics of the proposed *first-order* fuzzy partial logic $\mathrm{L}\forall^*$ is defined in a standard manner, the principal difference from the first-order fuzzy logic $\mathrm{L}\forall$ being the evaluation of predicate symbols in L^*-algebras instead of L-algebras.

The only non-straightforward task is then to extend the Tarski conditions for the basic (lattice) quantifiers of L$\forall$ to the extra truth value $*$. Like in the case of fuzzy partial connectives, this can be done in several meaningful ways, including Bochvar-style $(\forall, \exists)$, Sobociński-style $(\hat{\forall}, \hat{\exists})$, Kleene-style $(\bar{\forall}, \bar{\exists})$, etc. In the proposed first-order fuzzy partial logic, all of these variants are definable from $\forall$ and $\hat{\exists}$, which can thus be taken as primitive quantifiers of L$\forall^*$. Since the propositional fuzzy partial logic L* turns out to be implicative w.r.t. an ordering $\trianglelefteq$ of truth values in which $\forall$ and $\hat{\exists}$ are the lattice quantifiers, Rasiowa's quantifier axioms [6] for $\forall$ and $\hat{\exists}$ (taking the L*-definable connective $\trianglelefteq$ for implication in these axioms) provide a sound and complete axiomatic system for (the safe models [5] of) L$\forall^*$.

Russell-style *higher-order logic* over L* (denoted here by FPCT, for Fuzzy Partial Class Theory) can be defined analogously to Russell-style higher-order logic over L, also known as Fuzzy Class Theory (or FCT) [1]. Like in FCT, the language of FPCT contains variables for atomic elements, (fuzzy partial) classes of atomic elements, (fuzzy partial) classes of (fuzzy partial) classes of atomic elements, etc., with subsorts for tuples of all finite arities; the (crisp total) identity predicate $=$ on each sort; the (fuzzy partial) membership predicate $\in$ between successive sorts; and comprehension terms $\{x \mid \varphi\}$ of order $n+1$ for all well-typed formulas φ and variables x of order n. The intended models of FPCT consist of all L*-valued membership functions (representing fuzzy partial sets and relations) of all finite orders and arities over a given crisp set of elements. Henkin models of FPCT can be axiomatized in multi-sorted first-order L$\forall^*$ with crisp identity in a similar manner as FCT, namely by the axioms of extensionality, $(\forall x)(x \in A \equiv x \in B) \rightarrow A = B$, and typed comprehension, $y \in \{x \mid \varphi(x)\} \equiv \varphi(y)$, for all well-typed formulas φ and variables x, y of each order n (where $\equiv$ is the crisp defined connective of L* internalizing the identity of truth values and $\rightarrow$ is the Bochvar-style implication).

Since the intended models of FPCT consist of all (isomorphic representations of) partial L-valued membership functions (and the Henkin models contain at least all definable ones), FPCT is a suitable foundational framework for fuzzy partial set theory. An initial investigation of basic fuzzy partial set-theoretic notions (formalized in a simplified version of FPCT) has been sketched in [2]. Besides serving as a formal language for fuzzy partial set theory, the formal setting of FPCT makes it additionally possible to prove some meta-theoretical results on fuzzy partial set-theoretic notions. In particular, schematic theorems on elementary fuzzy partial set relations and operations (similar to those for FCT [1, Sect. 3.4]) can be proved for FPCT; this reduces a large part of elementary fuzzy partial set theory to propositional fuzzy partial logic L*.

Furthermore, since the horizontal level-cut representation of fuzzy partial sets in FPCT differs from that of fuzzy total sets in FCT only by the presence of the extra level for the truth value $*$, fuzzy partial sets can be represented by pairs $\langle X, A\rangle$, where X is a crisp total set (the $*$-level) and A is a fuzzy total set (represented by the remaining cuts). This provides the means for mutual formal syntactic interpretation between FCT and FPCT. In view of this mutual interpretability, predicate fuzzy partial logic can be understood as providing predefined relations and operations acting simultaneously on the crisp domains of fuzzy partial sets and their L-valued membership functions.

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References

[1] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[2] L. Běhounek and M. Daňková. Towards fuzzy partial set theory. Conference article (12 pages), submitted.

[3] L. Běhounek and V. Novák. Towards fuzzy partial logic. In *Proceedings of the IEEE 45th International Symposium on Multiple-Valued Logics (ISMVL 2015)*, pages 139–144, 2015.

[4] S. Blamey. Partial logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic (2nd Edition)*, volume 16, pages 261–353. Springer, 2011.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[6] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.