Towards a proof theory for fuzzy quantifiers: a calculus for rational Łukasiewicz logic and related systems

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Abstract

We introduce hypersequent calculi for Rational Łukasiewicz logic and related systems, motivated by investigations on fuzzy quantifiers.

Extended Abstract

Fuzzy quantifiers, i.e. expressions such as *many, few, about half*, are ubiquitous in natural language, though not easily treatable by means of classical logic. Finding formal models for these quantifiers within the framework of Mathematical Fuzzy Logic is a topic of crucial importance for the field, both for theoretical and application purposes, see e.g. [6]. Recent works [3, 5] have shown that a suitable game semantics framework can provide guiding principles for the characterization and systematic introduction of fuzzy quantifiers over Łukasiewicz logic. More precisely, families of fuzzy quantifiers can be defined in Łukasiewicz logic by extending the latter with an additional *random witness* quantifier Πx. We call the resulting system Ł(Π) in the following. The quantifier Πx is characterized by extending the so-called Giles-style dialogue games for Łukasiewicz logic (see e.g. [2]) by the following rule [3]:

(R_Π) If the current formula is $\Pi x F(x)$ then an element d from the domain D is chosen randomly and the game continues with $F(d)$.

The corresponding truth function for the formula $\Pi x F(x)$ is provided by the *expected value*. Taking a finite domain D and a uniform probability distribution, the truth function for $\Pi x F(x)$ amounts thus to the average of the truth values assumed by the predicate F over the elements of D, i.e.

$$v(\Pi x F(x)) = \sum_{d \in D} \frac{v(F(d))}{|D|}.$$

(for simplicity, we have identified above constants of the language and elements of the domain). So far, the quantifier Πx has only been investigated by semantics means: no axiomatization or proof-theoretic system for Ł(Π) is known. As a first step towards the proof-theoretic investigation of Ł(Π), in this contribution we present a cut-free hypersequent calculus for *Rational Łukasiewicz logic* RŁ, see e.g. [1]. The logic RŁ is an expansion of Ł with a family of unary connectives $\{\delta_n\}_{n\in N}$, standing for *division* operators. In other words, the intended evaluation v over the real interval $[0,1]$ of a formula $\delta_n\alpha$ is defined by $v(\delta_n\alpha) = v(\alpha)/n$ where $/$ stands for the usual division. Our calculus, which extends the one for Łukasiewicz logic introduced in [7], is shown to be sound and complete for the corresponding standard semantics. The calculus is not only interesting for the study of RŁ on its own, but is also relevant for further proof-theoretic investigations of Ł(Π): indeed, in RŁ connectives computing the average of truth values are definable. This kind of connectives can be seen in a sense as propositional counterparts of the quantifier Πx. Consider, for instance the connective π, introduced in [4] by game-theoretic ideas similar to those for the quantifier Π. The connective is characterized by the rule:

(R_π) If the current formula is $\alpha\pi\beta$ then a uniformly random choice determines whether the game continues with α or with β.

The corresponding truth function $v(\alpha\pi\beta) = (v(\alpha) + v(\beta))/2$ is clearly definable in RŁ, but not in Łukasiewicz logic. We can provide thus hypersequent calculi also for fragments of RŁ involving connectives such as π. In particular, we obtain a hypersequent calculus for the logic $KZ(\pi)$ [4], i.e. the expansion of the $\{\wedge, \vee, \neg\}$-fragment of Ł (also known as Kleene-Zadeh logic) with the connective π.

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