Cut Elimination for Gödel Logic with an operator adding a constant

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Gödel Logic extended by $\circ$

$\langle J, c\rangle$, $c > 0$

$$
\begin{array}{l}
J(\bot) = 0 \quad J(\top) = 1 \\
J(A) \in [0,1] \quad A \text{ atomic} \\
J(A \wedge B) = min\{J(A), J(B)\} \\
J(A \vee B) = max\{J(A), J(B)\} \\
J(A \supset B) = \begin{cases} 1 & J(A) \leq J(B) \\ J(B) & \text{else} \end{cases} \\
J(\circ A) = min\{J(A) + c, 1\}
\end{array}
$$

$$\neg A \sim A \supset \bot \quad A < B \sim (B \supset A) \supset B$$

Therefore

$$J(\neg A) = \begin{cases} 0 & J(A) > 0 \\ 1 & \text{else} \end{cases}$$

$$J(A < B) = \begin{cases} 1 & J(A) < J(B) \\ min\{J(A), J(B)\} & \text{else} \end{cases}$$

$G_0 = \{A | \models A\}$

A Hilbert type system for G_0

Minimal intuitionistic axiom schemata

$$+$$

$$\bot \supset A$$

$$+$$

$$A < T$$

$$+$$

$$A \supset B \vee B \supset A$$

$$+$$

$$A < \circ A$$

$$+$$

$$\circ(A \supset B) \leftrightarrow (\circ A \supset \circ B)$$

$$+$$

$$\frac{A \qquad A \supset B}{B}$$

G_0 is not compact

Let $\Gamma = \{A \supset B, \circ A \supset B, \circ \circ A \supset B, ...\}$

$$\Gamma \models B$$

but

$$\Delta \not\models B$$

for any finite subset $\Delta \subset \Gamma$

$J(\forall x A(x)) = inf(J(A(d))|d \in D_j)$,
$J(\exists x A(x)) = sup(J(A(d))|d \in D_j)$
F.O. G_0 is not r.e.
Let A be a F.O. formula s.t. all atoms A' are double negated, all quantifiers are relativized to R, let $P_1, ..., P_k$ be the predicates in A
$F_A \sim$

$$\forall x \neg\neg E(x,x) \wedge \forall x, y(\neg\neg E(x,y) \supset \neg\neg E(y,x))$$

$$\wedge \forall x, y, z(\neg\neg E(x,y) \wedge \neg\neg E(y,z) \supset \neg\neg E(x,z))$$

$$\wedge \forall x_1, ..., x_n, y_1, ..., y_n(\bigwedge \neg\neg E(x_i, y_i) \supset \neg\neg P_j(x_1, ..., x_n) \leftrightarrow P_j(y_1, ..., y_n))$$

$$\wedge \forall x, y(\neg E(x,y) \supset ((\circ R(x) \supset R(y)) \vee (\circ R(y) \supset R(x)))$$

$$H_A \sim F_A \supset A \vee \exists x R(x)$$

F.O. $G_0 \models H_A$
$\Leftrightarrow$ A is classically valid in all finite domains

Baaz / Fasching
Monotone operators on Gödel logics
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Chain on $\{X_1, ..., X_n\}$ C
$\bot \lhd_1 X_{i_1} \lhd_2 X_{i_2} ... X_{i_n} \lhd_{n+1} \top$
$\{X_1, ..., X_n\} = \{X_{i_1}, ..., X_{i_1}\}$, $\lhd_j \in \{\leftrightarrow, <\}$
Chain normal form
$\bigvee C \wedge \Psi_C(A) \quad \Psi_C(A) \in \{\bot, X_1, ..., X_n, \top\}$
ex for $A \vee \neg A$

$$(\bot \leftrightarrow A) \wedge (A < \top) \wedge \top$$

$$\vee(\bot < A) \wedge (A < \top) \wedge \bot$$

$$\vee(\bot < A) \wedge (A \leftrightarrow \top) \wedge \top$$

Consider chains on $\{\circ^k X_L | 1 \leq k \leq m, 1 \leq l \leq n\}$

Sequents of relation calculi

Standard Gödel logic

$A_1 \lhd_1 B_1 \mid \mid A_n \lhd_n B_n \quad \lhd_i \in \{<, \leq\}$

$$J(A < B) = \begin{cases} 1 & J(A) < J(B) \\ 0 & \text{else} \end{cases}$$

$$J(A \leq B) = \begin{cases} 1 & J(A) \leq J(B) \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} A \supset B \leq C \quad & \Leftrightarrow A \leq B \wedge 1 \leq C \vee B < A \wedge B \leq C \\ & \Leftrightarrow (A \leq B \vee B < A) \wedge (A \leq B \vee B \leq C) \\ & \wedge (1 \leq C \vee B < A) \wedge (1 \leq C \vee B \leq C) \\ & \Leftrightarrow (1 \leq C \vee B < A) \wedge (B \leq C) \end{aligned}$$

$$\frac{H|1 \leq C|B < A \qquad H|B \leq C}{H|A \supset B \leq C}$$

Cut

$$\frac{H | A < B \qquad H | B \leq A}{H}$$

Derives transitive cuts e.g.

$$\frac{H | A \leq B \qquad H | B \leq C}{H | A \leq C}$$

$$\frac{\dfrac{H | A \leq B}{H | A \leq C | C < B} \qquad H | B \leq C}{H | A \leq C}$$

The calculus RG_0^- for G_0

$A_1 \lhd_1 B_1 \mid \mid A_n \lhd_n B_n \quad \lhd_i \in \{\leq, <\}$
$J(A \leq B) = J(A \supset B) \quad J(A < B) = J((B \supset A) \supset B)$
Axioms A_1 $A \leq A$, A_2 $0 \leq A$, A_3 $A < 1$

External structural rules: weakening, contraction, exchange of sequents in the hypersequent
Communication

$$\frac{H|A \lhd_1 B \qquad H|C \lhd_2 D}{H|A \lhd_3 D|C \lhd_4 B}$$

$\lhd_1 = \lhd_2 = \leq \Rightarrow \{\lhd_3, \lhd_4\} = \{<, \leq\}$
$\{\lhd_1, \lhd_2\} = \{<, \leq\} \Rightarrow \lhd_3 = \lhd_4 = <$

Internal structural rules

$$\frac{H|A < B}{H|A \leq B}$$

$$\frac{H|A \lhd_1 B}{H|A \lhd_2 C|C \lhd_3 B}$$

$\lhd_1 = \leq \Rightarrow \{\lhd_2, \lhd_3\} = \{<, \leq\}$
$\lhd_1 = < \Rightarrow \{\lhd_2, \lhd_3\} = \{<\}$

CUT

$$\frac{H|A \leq B \qquad H|B \leq C}{H|A \leq C}$$

$$\frac{H|A < B \qquad H|B \leq C}{H|A < C}$$

$$\frac{H|A \leq B \qquad H|B < C}{H|A < C}$$

$$\frac{H|A < B \qquad H|B < C}{H|A < C}$$

Logical rules

$$\frac{H|C \lhd A \qquad H|C \lhd B}{H|C \lhd A \wedge B} \qquad \frac{H|A \lhd C|B \lhd C}{H|A \wedge B \lhd C}$$

$$\frac{H|C \lhd A|C \lhd B}{H|C \lhd A \vee B} \qquad \frac{H|A \lhd C \qquad H|B \lhd C}{H|A \vee B \lhd C}$$

$$\lhd \in \{<, \leq\}$$

$$\frac{H|A \leq B|C < B}{H|C < A \supset B} \qquad \frac{H|1 < C|B < A \qquad H|B < C}{H|A \supset B < C}$$

$$\frac{H|A \leq B|C \leq B}{H|C < A \supset B} \qquad \frac{H|1 \leq C|B < A \qquad H|B \leq C}{H|A \supset B \leq C}$$

Logical rules continued

$$\frac{H|A \leq B}{H|A < \circ B}$$

$$\frac{H|A \leq B}{H| \circ A \leq \circ B}$$

$$\frac{H|A < B}{H| \circ A < \circ B}$$

$$
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\dfrac{A \leq A}{A \leq B \mid B < A}
}{A \leq B \mid 1 \leq B \mid B < A}
}{1 \leq A \supset B \mid B < A}
}{1 \leq \circ(A \supset B) \mid B < A}
}{1 \leq \circ(A \supset B) \mid \circ B < \circ A}
\qquad
\dfrac{
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\dfrac{B \leq B}{B < A \mid A \leq B}
}{B \leq A \mid A \leq B}
}{B \leq A \supset B}
}{1 \leq \circ(A \supset B) \mid B \leq A \supset B}
}{1 \leq \circ(A \supset B) \mid \circ B \leq \circ(A \supset B)}
}{
\dfrac{\circ A \supset \circ B \leq \circ(A \supset B) \mid 1 \leq \circ(A \supset B)}{1 \leq (\circ A \supset \circ B) \supset \circ(A \supset B)}
}
$$

Lemma:

1. $RG_0^- \vdash A \leq B | A \supset B \leq B$
2. $RG_0^- \vdash H | 1 \leq A \vee B \Leftrightarrow RG_0^- \vdash H | 1 \leq A | 1 \leq B$
3. $RG_0^- \vdash H | 1 \leq A \supset B \Leftrightarrow RG_0^- \vdash H | A \leq B$
4. $RG_0^- \vdash H | 1 \leq A < B \Leftrightarrow RG_0^- \vdash H | A < B$

Theorem

For every hypersequent H

$$G_0 \models H \Leftrightarrow RG_0^- \vdash H$$

Lemma

For every formula A

$$G_0 \models A \Leftrightarrow RG_0^- \vdash 1 \leq A$$

Theorem

R_0^- admits cut-elimination.

Let $R_0 \sim R_0^- +$

$$\frac{H|A < A}{H|1 \leq A}$$

Lemma

R_0 admits cut-elimination

$$\dfrac{\dfrac{H|A \le C}{H|A \le B|B < C} \qquad H|B \le D}{H|A \le D|B < C}$$

$$\Downarrow$$

$$\dfrac{H|A \le C \qquad H|B \le D}{H|A \le D|B < C}$$