Right topologizing filters on commutative rings

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Substructural logics are modeled by residuated structures [5], although the latter have independent origins in the context of the theory of ideals of a commutative ring in algebra. A lattice ordered monoid is a monoid endowed with a lattice order that is compatible with the monoid operation in the sense that the latter distributes over finite joins. The systematic study of these structures which has its origin in the 1930s paper of Ward and Dilworth [4], [3](see also [1] and [2]), was motivated by a classical prototype, the set of ideals of a commutative ring with identity; the monoid operation in this instance being ideal multiplication with the lattice operation derived from the usual partial order.

The set of ideals of an arbitrary (not necessarily commutative) ring R, which we shall denote by IdR, is a lattice ordered monoid which is in general noncommutative (by which is meant the monoid operation is not commutative), but which possesses the right and left residuation property: if I and J are ideals of R, then there is (a unique) largest ideal K of R, given by $K = \{r \in R : Jr \subseteq I\}$, such that $JK \subseteq I$. In this situation K is called the right residual of I by J and denoted by $J^{-1}I$. The left residual of I by J denoted by IJ^{-1}, is the largest ideal K of R such that $KJ \subseteq I$ and comprises $\{r \in R : rJ \subseteq I\}$.

Given its commutative ring theoretic antecedents, early work on lattice ordered monoids focused on the development of commutative residuation theory. However residuated structures later found application in different areas of mathematics, in particular in the algebra of binary relations and the model theory of nonclassical logics.

Residuated lattice ordered monoids also arise in torsion theory, for the set of right topologizing filters on an arbitrary ring R (or equivalently, the collection of all hereditary pretorsion classes of right R-modules) is the order dual of a lattice ordered monoid. It is the study of these structures in this torsion theoretic context that shall be the setting of this paper.

A right topologizing filter on a ring R is a nonempty family $\mathfrak{F}$ of right ideals of a ring R that satisfies the following three conditions:

F1. $A \in \mathfrak{F}$ implies $B \in \mathfrak{F}$ whenever B is a right ideal of R containing A;

F2. $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$;

F3. $A \in \mathfrak{F}$ and $r \in R$ implies $r^{-1}A \stackrel{\text{def}}{=} \{x \in R \,:\, rx \in A\} \in \mathfrak{F}$.

The set $FilR_R$ of all right topologizing filters on a fixed but arbitrary ring R is both a complete lattice under inclusion, and a monoid with respect to an order compatible, but in general noncommutative binary operation :. It is known that the order dual $[FilR_R]^{du}$ of $FilR_R$ is always left residuated, meaning, for each pair $\mathfrak{F}, \mathfrak{G} \in FilR_R$ there exists a smallest filter $\mathfrak{H} \in FilR_R$ such that $\mathfrak{H} : \mathfrak{G} \supseteq \mathfrak{F}$, but is not, in general, right residuated (there exists a smallest filter $\mathfrak{H}$ such that $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$).
The binary operation : is defined on $[FilR_R]^{du}$ as follows:

$$\mathfrak{F} : \mathfrak{G} \stackrel{\text{def}}{=} \{K \leq R_R : \exists H \in \mathfrak{F} \text{ such that } K \subseteq H \;\&\; h^{-1}K \in \mathfrak{G} \;\forall h \in H\}.$$

Thus the *order dual* $[\mathrm{Fil}R_R]^{du}$ of $\mathrm{Fil}R_R$ has the *structure of a lattice ordered monoid*.

The importance of the structure $\mathrm{Fil}R_R$ (as a tool for analysing the ring R), lies in the fact that it encodes at least as much information about the ring R as does the *ideal lattice* $\mathrm{Id}R$, for there is a *canonical structure preserving embedding* (that is in general not onto) of $\mathrm{Id}R$ into $[\mathrm{Fil}R_R]^{du}$ that takes each $I \in \mathrm{Id}R$ onto the *set of all right ideals* of R containing I. However, whereas $\mathrm{Id}R$ enjoys *residuation on both sides*, $[\mathrm{Fil}R_R]^{du}$, is in general, *left but not right residuated*.
It has been shown [6] that for every right fully bounded noetherian ring R, $[\mathrm{Fil}R_R]^{du}$ is two-sided residuated and that a valuation domain will be too if and only if it is rank one discrete.
If R is any ring for which (the monoid operation : on) $\mathrm{Fil}\,R_R$ is commutative, then obviously $[\mathrm{Fil}\,R_R]^{\mathrm{du}}$ is two-sided residuated.
The purpose of this paper is to show that the converse is true whenever the ring R is commutative. That is, if R is a commutative ring for which $[\mathrm{Fil}\,R_R]^{\mathrm{du}}$ is *two-sided residuated*, then $\mathrm{Fil}\,R_R$ is commutative, that is to say, $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}$ $\forall \mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\,R_R$. We also provide several non-torsion theoretic characterizations of the two-sided residuated property for a commutative ring.

References

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