Two-sided residuation in a commutative ring

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Outline

Right topologizing filters on a ring

Throughout this talk:

- R is an associative ring with unity and M is a right R-module.
- $\mathrm{Id}\, R \stackrel{\mathrm{def}}{=} \{$(two-sided) ideals of $R\}$.

Definition. A nonempty family $\mathfrak{F}$ of right ideals of a ring R is called a right topologizing filter on R if:

F1. $I \in \mathfrak{F} \Rightarrow$ any right ideal $J \supseteq I$ belongs to $\mathfrak{F}$;

F2. $I, J \in \mathfrak{F}$ implies $I \cap J \in \mathfrak{F}$;

F3. $I \in \mathfrak{F}$ and $r \in R$ implies $r^{-1}I \stackrel{\mathrm{def}}{=} \{x \in R : rx \in I\} \in \mathfrak{F}$.

- $\mathrm{Fil}\, R_R \stackrel{\mathrm{def}}{=} \{$set of all right topologizing filters on $R\}$.

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$\mathrm{Fil}\, R_R$ admits a binary operation : defined by

$$\mathfrak{F} : \mathfrak{G} \stackrel{\text{def}}{=} \{K \leq R_R \;:\; \exists H \in \mathfrak{F} \text{ s.t. } K \subseteq H \;\&\; h^{-1}K \in \mathfrak{G}\; \forall h \in H\}.$$

- We say $[\mathrm{Fil}\, R_R]^{du}$ is left (resp. right) residuated if given $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$ there exists a smallest filter $\mathfrak{H}$ in $\mathrm{Fil}\, R_R$ satisfying $\mathfrak{H} : \mathfrak{G} \supseteq \mathfrak{F}$ (resp. $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$).
- We denote the left (resp. right) residual of $\mathfrak{F}$ by $\mathfrak{G}$ $\mathfrak{F}\mathfrak{G}^{-1}$ (resp. $\mathfrak{G}^{-1}\mathfrak{F}$).

Theorem.[Golan, Proposition 4.1, p. 43] $\langle \mathrm{Fil}\, R_R; \subseteq^{du}; : \rangle$ is a lattice ordered, left residuated monoid but not in general right residuated. ■

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For any ring R $\mathrm{Fil}\, R_R$ is larger than $\mathrm{Id}\, R$ for *Id R* embeds as a sublattice in the dual of $\mathrm{Fil}\, R_R$. That is:

$$\begin{array}{ccc} \mathrm{Id}\, R & \underset{\text{order reversing}}{\hookrightarrow} & [\mathrm{Fil}\, R_R]^{du} \\ I & \longmapsto & \eta(I) \stackrel{\text{def}}{=} \{K \leq R_R \,:\, K \supseteq I\} \end{array}$$

Question. When is $\langle \mathrm{Fil}\, R_R, \subseteq^{du}, :\rangle$ both right and left residuated?

Theorem. [Beachy and Blair, 1978] The following statements are equivalent for a ring R:

- R is right artinian (satisfies DCC on right ideals);
- $\mathrm{Fil}\, R_R = \mathrm{Id} R_R$, i.e., every $\mathscr{F} \in \mathrm{Fil}\, R_R$ has the form $\mathscr{F} = \eta(I)$, for some two-sided ideal I of R.

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Theorem. [van den Berg, 1999] If R is a commutative noetherian ring then $FilR_R$ is commutative and hence $[\,Fil\ R_R]^{du}$ is two-sided residuated.

Question. If R is an arbitrary right noetherian ring, is $\mathrm{Fil}\ R_R$ left and right residuated?

Theorem. Partial answer [Nega Arega, van den Berg, 2016] If R is a right fully bounded right noetherian ring, then $\mathrm{Fil}\ R_R$ is left and right residuated.

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Two-sided residuation in a commutative ring

Notation. We use the following abbreviation throughout this section.

- DDF will denote downward directed family.
- UDF will denote upward directed family.
- AI(s) will denote annihilator ideal(s).
- AS(s) will denote annihilator submodule(s).
- $I \lhd R$ will denote proper ideal of R.

Theorem 3.1. [van den Berg, 1999] For any ring R, if : is commutative then $\mathrm{Fil}\, R_R$ is two-sided residuated.

- Let P be a poset. $X \subseteq P$ is said to be downward (upward) directed if, given any pair x_1, x_2 in X, $\exists y \in X$ s.t $x_1 \geq y$ and $x_2 \geq y$ ($x_1 \leq y$ and $x_2 \leq y$).

Proposition 3.2. Let R be a commutative ring for which $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated. If I is an arbitrary ideal of R, then $\eta(I)$ is central in $\mathrm{Fil}\, R$.

Theorem 3.4. Let R be a commutative ring for which $[\mathrm{Fil}\, R_R]^{du}$ is two-sided residuated. Then $\mathrm{Fil}\, R_R$ is commutative, i.e., $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}\ \forall \mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$.

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(a) $\mathrm{Fil}\, R_R$ is commutative;

(b) $[\, Fil\, R_R]^{du}$ is two-sided residuated;

(c) The ring R/I satisfies the ACC on AIs for all $I \lhd R$;

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(e) $(R/I)_R$ satisfies the ACC on ASs for all $I \lhd R$;

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Semiartinian modules

- An R-module M is semiartinian if and only if every non-zero factor module of M has nonzero socle.

Theorem 3.6. The fsae for a commutative semiartinian ring R:

(a) R is artinian;

(b) $\mathrm{soc}\,(R_R)$ and $\mathrm{soc}^2\,(R_R)$ are both f.g;

(c) $\mathrm{Fil}\,R$ is commutative and $\mathrm{soc}\,(R_R)$ is f.g.

Question. Is artinianness a necessary condition for $\mathrm{Fil}\,R$ to be commutative? The answer is No.

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(c) $\mathrm{Fil}\,R$ is commutative and $\mathrm{soc}\,(R_R)$ is f.g.

Question. Is artinianness a necessary condition for $\mathrm{Fil}\,R$ to be commutative? The answer is No.

Semiartinian modules

- An R-module M is semiartinian if and only if every non-zero factor module of M has nonzero socle.

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A class of examples of commutative semiartinian rings

- Let $R = \langle F, V, U, \mu \rangle$ where F is a field, V and U F-spaces, and $\mu : V \times V \to U$ a symmetric F-bilinear map with $\mu(v, v') = v \cdot v'$.

- We equip the set $F \times V \times U$ with an F-algebra structure by taking addition to be natural and defining multiplication by:

$$(a_1, v_1, u_1) \cdot (a_2, v_2, u_2) \stackrel{\text{def}}{=} (a_1 a_2, a_1 v_2 + a_2 v_1, a_1 u_2 + v_1 \cdot v_2 + a_2 u_1).$$

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Continue...

Alternatively:

$$\begin{pmatrix} a_1 & v_1 & u_1 \\ 0 & a_1 & v_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 & u_2 \\ 0 & a_2 & v_2 \\ 0 & 0 & a_2 \end{pmatrix} =$$

$$\begin{pmatrix} a_1 a_2 & a_1 v_2 + a_2 v_1 & a_1 u_2 + v_1 . v_2 + a_2 u_1 \\ 0 & a_1 a_2 & a_1 v_2 + a_2 v_1 \\ 0 & 0 & a_1 a_2 \end{pmatrix}.$$

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- The symmetry of μ guarantees that R is commutative.
- $\operatorname{soc}(R_R) = \langle 0, 0, U \rangle = \{(0, 0, u) : u \in U\}$ and
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- If X and Y are F-subspaces of V and U, respectively, we define F-subspace $X^{-1}Y$ of V as follows:

$$X^{-1}Y \stackrel{\text{def}}{=} \{v \in V : X \cdot v \subseteq Y\} = \{v \in V : x \cdot v \in Y \ \forall x \in X\}$$

Lemma 3.7. The following assertions are equivalent for a proper nonzero ideal A of $R = \langle F, V, U, \mu \rangle$:

(a) $A = K^{-1}I = \{r \in R : Kr \subseteq I\}$ for some $I, K \in \mathrm{Id}\, R$;

(b) $A = \langle 0, X^{-1}Y, U \rangle$ for some F-subspaces $X$and $Y$of $V$and U respectively.

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Theorem 3.8. Let $R = \langle F, V, U, \mu \rangle$. The fsae:

(a) $\mathrm{Fil}\, R$ is commutative;

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Corollary 3.9. Let $R = \langle F, V, U, \mu \rangle$. If $\dim_F V$ is finite, then $\mathrm{Fil}\, R$ is commutative.

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- Commutative non-artinian semiartinian rings R for which $\mathrm{Fil}\, R$ is commutative.

Example 3.10. We choose V, U and μ such that $\dim_F V$ and $\dim_F U$ are both infinite, but the ring $R = \langle F, V, U, \mu \rangle$ is such that $\mathrm{Fil}\, R$ is commutative.

Let T be the commutative F-algebra defined by

$$T \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & w \\ 0 & a \end{pmatrix} : a \in F, w \in W \right\}$$

where W is any infinite dimensional F-space.

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Thank you!