A uniform way to build symmetric DL-algebras via Boolean algebras and prelinear semihoops

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The variety $\mathbb{DL}$ of DL-algebras is the class of those MTL-algebras satisfying the following equations

$$\neg\neg(\neg\neg x \to x) = 1, \qquad \text{and} \qquad (2x)^2 = 2(x^2)\,,$$

where, as usual, $\neg x := x \to 0$, $x \oplus y := \neg(\neg x * \neg y)$, $x^2 := x * x$ and $2x := x \oplus x$.

Let us define the variety $\mathbb{SDL}$ of *symmetric DL-algebras* as the subvariety of $\mathbb{DL}$ obtained by the following equation

$$\neg(x^2) \to (\neg\neg x \to x) = 1. \tag{1}$$

In order to explain what algebras are captured by (1) recall that, for any MTL-algebra $\mathbf{A}$, its *radical* (denoted $Rad(\mathbf{A})$) is the intersection of all its maximal filters. While the *co-radical* of $\mathbf{A}$ is defined, following [1], as $coRad(\mathbf{A}) = \{x \in A \mid \neg x \in Rad(\mathbf{A})\}$. Thus, (1) characterizes all those DL-algebras for which the elements of the co-radical are involutive. Equivalently, for every $\mathbf{A} \in \mathbb{SDL}$, one has $coRad(\mathbf{A}) = \{\neg x \mid x \in Rad(\mathbf{A})\}$.

Relevant subvarieties of $\mathbb{SDL}$ are the variety $\mathbb{G}$ of Gödel algebras, the variety $\mathbb{P}$ of Product algebras and the variety $\mathbb{DLMV}$ generated by perfect MV-algebras.

In [3] the authors provided a categorical equivalence between the category $\mathbb{P}$ of product algebras with homomorphisms, and a category $\mathcal{T}_C$ whose objects are triples $(\mathbf{B}, \mathbf{C}, \vee_e)$ where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop and $\vee_e : B \times C \to C$ is a suitably defined map, intuitively representing the natural join between Boolean and cancellative elements. If $(\mathbf{B}, \mathbf{C}, \vee_e)$ and $(\mathbf{B}', \mathbf{C}', \vee'_e)$ are two triples, a morphism in $\mathcal{T}_C$ is a pair (f, g) where $f : \mathbf{B} \to \mathbf{B}'$ is a Boolean homomorphism, $g : \mathbf{C} \to \mathbf{C}'$ is a hoop homomorphism, and for every $(b, c) \in B \times C$, $g(b \vee_e c) = f(b) \vee'_e g(c)$. These pairs (f, g) have been called *good morphisms pairs* in [3].

Given a triple $(\mathbf{B}, \mathbf{C}, \vee_e)$ in which $\mathbf{B}$ is the two element Boolean algebra $\mathbf{2}$, the resulting product algebra $\mathbf{P}$ is directly indecomposable and, indeed, every directly indecomposable product algebra arises in this way.

From [1] we can derive that every directly indecomposable SDL-algebra $\mathbf{A}$ can be constructed starting from a prelinear semihoop $\mathbf{H}$ and by a *dl-admissible* map $\delta : H \to H$ (see [1] for further details).

The main aim of our work is to generalise the ideas and constructions used in [1, 3] and provide a uniform approach to establish categorical equivalences between relevant subcategories of SDL-algebras, and between those subcategories and categories generalising $\mathcal{T}_C$ of [3].

Triples with prelinear semihoops

Let $\mathbb{H}$ be any subvariety of the variety $\mathbb{PSH}$ of prelinear semihoops and let $\mathcal{T}_{\mathbb{H}}$ be the category whose objects are triples $(\mathbf{B}, \mathbf{H}, \vee_e)$ where $\mathbf{B}$ and $\vee_e$ are as above, while $\mathbf{H}$ is a prelinear semihoop in $\mathbb{H}$. Morphisms of $\mathcal{T}_{\mathbb{H}}$ are good morphisms pairs. For every triple $(\mathbf{B}, \mathbf{H}, \vee_e)$ and for every dl-admissible operator $\delta : H \to H$ we can now define the algebra $\mathbf{B} \otimes_e^{\delta} \mathbf{H}$ by suitably generalising the construction of $\mathbf{B} \otimes_e \mathbf{H}$ provided in [3]. Every algebra $\mathbf{B} \otimes_e^{\delta} \mathbf{H}$ is a SDL-algebra. In particular, if we choose $\delta_L : x \in H \mapsto 1 \in H$ as admissible operator, $\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is an SMTL-algebra, while if we choose the identity map $\delta_D : x \in H \mapsto x \in H$, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is an IDL-algebra (i.e. an *involutive* DL-algebra). Let us denote, for every $\mathbb{H}$ subvariety of $\mathbb{PSH}$, $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathbb{IDL}_{\mathbb{H}}$ the full subcategories of respectively SMTL and IDL-algebras such that each $\mathbf{A} \in \mathbb{SMTL}_{\mathbb{H}} \cup \mathbb{IDL}_{\mathbb{H}}$ has its largest prelinear sub-semihoop in $\mathbb{H}$. We prove the following.

Theorem 0.1. *For every subvariety $\mathbb{H}$ of $\mathbb{PSH}$, the category $\mathcal{T}_{\mathbb{H}}$ is equivalent to $\mathbb{SMTL}_{\mathbb{H}}$ and to $\mathbb{IDL}_{\mathbb{H}}$. Hence, in particular, $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathbb{IDL}_{\mathbb{H}}$ are equivalent categories for every $\mathbb{H}$.*

The following are relevant examples of varieties which are categorically equivalent by consequence of Theorem 0.1 above.

(i) The variety $\mathbb{SMTL}$ of SMTL-algebras and the variety $\mathbb{IDL}$ of involutive DL-algebras, which in turn are equivalent to the category $\mathcal{T}_{\mathbb{PSH}}$.
(ii) The variety $\mathbb{P}$ of product algebras and the variety $\mathbb{DLMV}$ generated by perfect MV-algebras, which in turn are equivalent to the category $\mathcal{T}_{\mathbb{CH}}$ where $\mathbb{CH}$ is the variety of cancellative hoops.
(iii) The variety $\mathbb{G}$ of Gödel algebras and the variety $\mathbb{NM}^-$ of Nilpotent Minimum algebras generated by chains with no negation fixpoint, which in turn are equivalent to the category $\mathcal{T}_{\mathbb{GH}}$, where $\mathbb{GH}$ is the variety of Gödel hoops.

From triples to quadruples

To construct all algebras in $\mathbb{SDL}$ we shall use all dl-admissible operators. In order to cope with generic dl-admissible operators, we take another step of generalisation and introduce prelinear-semihoop-based *quadruples*. These are defined in the following way. Fix again any subvariety $\mathbb{H}$ of $\mathbb{PSH}$ and let $\mathcal{Q}_{\mathbb{H}}$ be the following category:

- The objects of $\mathcal{Q}_{\mathbb{H}}$ are quadruples $(\mathbf{B}, \mathbf{H}, \vee_e, \delta)$ where $\mathbf{H} \in \mathbb{H}$, $(\mathbf{B}, \mathbf{H}, \vee_e) \in \mathcal{T}_{\mathbb{H}}$ and $\delta : H \to H$ is dl-admissible.
- The morphisms are pairs $(f, g) : (\mathbf{B_1}, \mathbf{H_1}, \vee_e^1, \delta_1) \to (\mathbf{B_2}, \mathbf{H_2}, \vee_e^2, \delta_2)$, such that (f, g) is a good morphism pair from $(\mathbf{B_1}, \mathbf{H_1}, \vee_e^1)$ to $(\mathbf{B_2}, \mathbf{H_2}, \vee_e^2)$, and $g(\delta_1(x)) = \delta_2(g(x))$ for all $x \in H_1$.

Let $\mathbb{SDL}_{\mathbb{H}}$ be the full subcategory of $\mathbb{SDL}$ consisting of those algebras whose largest prelinear sub-semihoop belongs to $\mathbb{H}$. Then the following holds.

Theorem 0.2. *Given any subvariety $\mathbb{H}$ of $\mathbb{PSH}$, the categories $\mathbb{SDL}_{\mathbb{H}}$ and $\mathcal{Q}_{\mathbb{H}}$ are equivalent. In particular $\mathcal{Q}_{\mathbb{PSH}}$ and $\mathbb{SDL}$ are equivalent categories.*

References

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