Equivalence of varieties of MTL-algebras built from prelinear semihoops

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MTL-algebras are systems $(A, *, \to, \wedge, \vee, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, $(A, *, 1)$ is a commutative monoid, and such that residuation $x * y \leq z \iff x \leq y \to z$, and prelinearity $(x \to y) \vee (y \to x) = 1$ hold for all $x, y, z \in A$. As is well known, MTL-algebras forms a variety, denoted $\mathbb{MTL}$.

In each MTL-algebra we define $\neg x := x \to 0$, $x \oplus y := \neg(\neg x * \neg y)$, $x^2 := x * x$ and $2x := x \oplus x$. Then the subvariety $\mathbb{DL}$ of $\mathbb{MTL}$ consists of those MTL-algebras A satisfying

$$\neg\neg(\neg\neg x \to x) = 1, \qquad \text{and} \qquad (2x)^2 = 2(x^2)\,,$$

for all $x \in A$ ([1]). We focus on two major subvarieties of $\mathbb{DL}$: the variety $\mathbb{SMTL}$ consisting of those $\mathbb{DL}$-algebras A such that

$$x \wedge \neg x = 0\,,$$

for all $x \in A$, and the variety $\mathbb{IDL}$ (also called $\mathbb{IBP}_0$) of the $\mathbb{DL}$-algebras A such that

$$\neg\neg x = x\,,$$

for all $x \in A$.

Relevant subvarieties of $\mathbb{SMTL}$ are the variety $\mathbb{G}$ of Gödel algebras, and the variety $\mathbb{P}$ of Product algebras, while relevant subvarieties of $\mathbb{IDL}$ are the variety $\mathbb{DLMV}$ generated by Chang's MV-algebra and the variety $\mathbb{NM}^-$ generated by the NM-algebra $[0, 1] \setminus \{1/2\}$, which is axiomatised by $\neg(x^2) \vee (x \to x^2) = 1$.

In this work we shall exhibit a categorical equivalence between $\mathbb{SMTL}$ and $\mathbb{IDL}$, translating one into the other through MTL-terms. We further find an isomorphism between the lattice of subvarieties of $\mathbb{SMTL}$ and the lattice of subvarieties of $\mathbb{IDL}$. To prove these correspondences we use *prelinear semihoops* as building blocks for DL-algebras (see also [3]). Recall that the variety $\mathbb{PSH}$ of *prelinear semihoops* consists exactly of the subreducts of MTL-algebras over the signature $(*, \to, \wedge, \vee, 1)$ ([2]). We show how to translate axioms of prelinear semihoops into axioms of SMTL-algebras and IDL-algebras, respectively.

Consider the following terms in the language of MTL-algebras:

$$\sigma(x) = \neg(\neg(x^2))^2 \wedge x\,,$$

$$\tau(x,y) = (x \wedge y) \vee (\neg x \wedge \neg y)\,.$$

Lemma 1. *For each $\mathbb{IDL}$-algebra A let*

$$S(A) = (\sigma(A), *, \Rightarrow, \wedge, \vee, 0, 1)\,,$$

where $x \Rightarrow y = \sigma(x \to y)$. Then $S(A)$ is a $\mathbb{SMTL}$-algebra.

Lemma 2. *For each $\mathbb{SMTL}$-algebra A let $i(A) = \{(b^-, b^+) \mid b^-, b^+ \in A,\ \tau(b^-, b^+) = 0\}$ and*

$$(b^-, b^+) \odot (c^-, c^+) = ((c^+ \to b^-) \wedge (b^+ \to c^-), b^+ * c^+)$$

$$(b^-, b^+) \Rightarrow (c^-, c^+) = (b^+ * c^-, (b^+ \to c^+) \wedge (c^- \to b^-))$$

$$(b^-, b^+) \sqcap (c^-, c^+) = (b^- \vee c^-, b^+ \wedge c^+)$$

$$(b^-, b^+) \sqcup (c^-, c^+) = (b^- \wedge c^-, b^+ \vee c^+)\,.$$

Then

$$I(A) = (i(A), \odot, \Rightarrow, \sqcap, \sqcup, (0,1), (1,0))\,,$$

is a $\mathbb{IDL}$-algebra.

Theorem 3. *If $f\colon B \to C$ is an homomorphism of $\mathbb{IDL}$-algebras then $S(f) = f \restriction S(B)$ is a homomorphism $S(f)\colon S(B) \to S(C)$.*

If $f\colon B \to C$ is an homomorphism of $\mathbb{SMTL}$-algebras then $I(f)\colon (b^-, b^+) \mapsto (f(b^-), f(b^+))$ is a homomorphism $I(f)\colon I(B) \to I(C)$.

The functors

$$S\colon \mathbb{IDL} \to \mathbb{SMTL} \qquad \textit{and} \qquad I\colon \mathbb{SMTL} \to \mathbb{IDL}$$

realise a categorical equivalence between $\mathbb{IDL}$ and $\mathbb{SMTL}$.

Given a subvariety $\mathbb{V}$ of $\mathbb{MTL}$, we write $\mathbb{V}_{d.i.}$ for the full subcategory of directly indecomposable $\mathbb{V}$-algebras.

Theorem 4.

$$\mathbb{SMTL}_{d.i.} \equiv \mathbb{PSH} \equiv \mathbb{IDL}_{d.i.}\,.$$

Given a formula $\varphi(x_1, \dots, x_n)$ in the language of prelinear semihoops, let its translation be the formula

$$t(\varphi) = \varphi \vee \bigvee_{i=1}^{n} \neg(x_i^2)\,.$$

Theorem 5. *Let* $\mathbb{W}$ *be a subvariety of* $\mathbb{PSH}$*, and assume* $\mathbb{W}$ *is axiomatised by a set of equations* $\{\varphi_i = 1\}_{i \in I}$*. Let*

$$S(\mathbb{W}) = \{A \in \mathbb{SMTL} \mid A \textit{ satisfies } t(\varphi_i) = 1 \textit{ for all } i \in I\}\,,$$

and

$$I(\mathbb{W}) = \{A \in \mathbb{IDL} \mid A \textit{ satisfies } t(\varphi_i) = 1 \textit{ for all } i \in I\}\,.$$

Then

$$S(\mathbb{W}) \equiv I(\mathbb{W})\,.$$

Moreover, the lattices of subvarieties of $\mathbb{SMTL}$*,* $\mathbb{IDL}$ *and* $\mathbb{PSH}$ *are isomorphic.*

References

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