Cut Elimination for Gödel Logic with an Operator adding a Constant (Abstract)*

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Propositional Gödel logic is an extension of intuitionistic logic that takes truth values in the set $[0,1]$. We study an extension of Gödel logic by an unary operator $\circ$ that adds a positive constant to truth values. The propositional fragment of this extension can be axiomatized by the following two formulae:

1. $A \prec \circ A$, and
2. $\circ(A \to B) \leftrightarrow (\circ A \to \circ B)$,

where $A \prec B$ by $(B \to A) \to B$.

Definition 1. *We consider the language $\mathcal{L}$ of propositional logic, augmented with a unary operator $\circ$. A propositional Gödel $\circ$-valuation $\mathfrak{I}$ is a function from the set of propositional variables into $[0,1]$ with $\mathfrak{I}(\bot) = 0$ and $\mathfrak{I}(\top) = 1$, together with a real number $c \in (0,1]$. This valuation can be extended to a function mapping formulas from $\mathcal{L}$ into $[0,1]$ as follows:*

$$\mathfrak{I}(A \wedge B) = \min\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \quad \mathfrak{I}(A \vee B) = \max\{\mathfrak{I}(A), \mathfrak{I}(B)\},$$

$$\mathfrak{I}(A \to B) = \begin{cases} \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B), \\ 1 & \text{if } \mathfrak{I}(A) \leq \mathfrak{I}(B), \end{cases}$$

$$\mathfrak{I}(\circ A) = \min\{\mathfrak{I}(A) + c, 1\}.$$

We define a sequents-of-relations calculus $\mathsf{RG}_\circ$. Herein a sequent is an expression

$$A_1 \lhd_1 B_1 | \dots | A_n \lhd_n B_n,$$

where the each symbol $\lhd_i$ is either $<$ or $\leq$ and plays a role similar to the sequent arrow in traditional sequent calculi. We say a component $A < B$ is satisfied by an interpretation $\mathfrak{I}$ if $\mathfrak{I}(A \prec B) = 1$ and a component $A \leq B$ is satisfied by an interpretation $\mathfrak{I}$ if $\mathfrak{I}(A \to B) = 1$. A sequent Σ is satisfied by $\mathfrak{I}$ if $\mathfrak{I}$ satisfies at

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least one of its components.

The axioms of $\mathsf{RG}_\circ$ are:

A1. $A \leq A$ A2. $0 \leq A$ A3. $A < 1$.

The external structural rules are:

$$\frac{\mathcal{H}|A < B|A < B}{\mathcal{H}|A < B}\ c_1 \qquad \frac{\mathcal{H}|A \leq B|A \leq B}{\mathcal{H}|A \leq B}\ c_2$$

$$\frac{\mathcal{H}}{\mathcal{H}|A < B}\ w_1 \qquad \frac{\mathcal{H}|A \lhd_1 B \qquad \mathcal{H}|C \lhd_3 D}{\mathcal{H}|A \lhd_3 D|C \lhd_4 B}\ com$$

where either $\lhd_1 = \lhd_2 = \leq$ and $\{\lhd_3, \lhd_4\} = \{<, \leq\}$, or $< \in \{\lhd_1, \lhd_2\}$ and $\lhd_3 = \lhd_4 = <$. The internal structural rules are

$$\frac{\mathcal{H}|A < B}{\mathcal{H}|A \leq B}\ w_2 \qquad \frac{\mathcal{H}|A \leq B}{\mathcal{H}|A < C|C \leq B}\ w_3$$

$$\frac{\mathcal{H}|A \leq B}{\mathcal{H}|A \leq C|C < B}\ w_4 \qquad \frac{\mathcal{H}|A < B}{\mathcal{H}|A < C|C < B}\ w_5$$

$$\frac{\mathcal{H}|A < B \qquad \mathcal{H}|B < C}{\mathcal{H}|A < C}\ cut_1 \qquad \frac{\mathcal{H}|A < B \qquad \mathcal{H}|B \leq C}{\mathcal{H}|A < C}\ cut_2$$

$$\frac{\mathcal{H}|A \leq B \qquad \mathcal{H}|B < C}{\mathcal{H}|A < C}\ cut_3 \qquad \frac{\mathcal{H}|A \leq B \qquad \mathcal{H}|B \leq C}{\mathcal{H}|A \leq C}\ cut_4$$

We proceed to logical inferences. The rules for conjunction and disjunction are obtained by replacing $\lhd$ by $<$ or $\leq$ in the following rules:

$$\frac{\mathcal{H}|C \lhd A \qquad \mathcal{H}|C \lhd B}{\mathcal{H}|C \lhd (A \wedge B)}\ \wedge_1^{\lhd} \qquad \frac{\mathcal{H}|A \lhd C|B \lhd C}{\mathcal{H}|(A \wedge B) \lhd C}\ \wedge_2^{\lhd}$$

$$\frac{\mathcal{H}|C \lhd A|C \lhd B}{\mathcal{H}|C \lhd (A \vee B)}\ \vee_1^{\lhd} \qquad \frac{\mathcal{H}|A \lhd C \qquad \mathcal{H}|B \lhd C}{\mathcal{H}|(A \vee B) \lhd C}\ \vee_2^{\lhd}$$

The rules for implication are:

$$\frac{\mathcal{H}|A \leq B \mid C < B}{\mathcal{H}|C < (A \to B)}\ \to_1 \qquad \frac{\mathcal{H}|1 < C|B < A \qquad \mathcal{H}|B < C}{\mathcal{H}|(A \to B) < C}\ \to_2$$

$$\frac{\mathcal{H}|A \leq B \mid C \leq B}{\mathcal{H}|C \leq (A \to B)} \to_3 \qquad \frac{\mathcal{H}|1 \leq C \mid B < A \qquad \mathcal{H}|B \leq C}{\mathcal{H}|(A \to B) \leq C} \to_4$$

Finally, the rules for the operator $\circ$ are as follows:

$$\frac{\mathcal{H}|A \leq B}{\mathcal{H}|A < \circ B} \circ_1 \qquad \frac{\mathcal{H}|A \leq B}{\mathcal{H}|\circ A \leq \circ B} \circ_2$$

Theorem 2. $\mathsf{RG}_\circ$ *is sound and complete for the intended interpretation and admits cut elimination.*